BSc. (Math) part-II
paper IV
Topic :- KEPLER'S LAWS

Q:- State and explain Kepler's law of planetary motion

Ans:- The following three laws of planetary motion round the sun are known as Kepler's laws:-

Kepler's first law

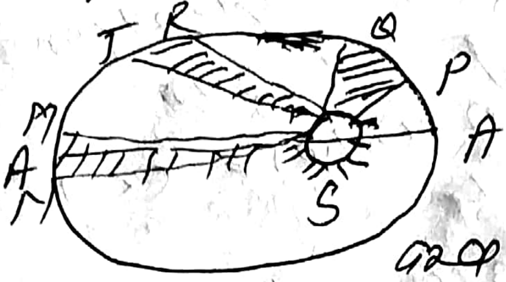
Each planet describes an ellipse having the sun in one of its foci.

Explanation of first law of Kepler:-

Let $ABA'B'$ be the orbit of planet P. The orbit is an ellipse. C is the centre of ellipse. The sun is at the focus S. AA' is the major axis and BB' is the minor axis of the ellipse. The point A, that is a point on the orbit of the planet, is nearest to the sun and is called the perihelion and the point A' which is also point on the orbit of the planet, is farthest from the sun and is called the aphelion.

Kepler's second law

The radius vector drawn from the centre of the sun to the sun to the planet sweeps over equal areas in equal times.



Explanation of the second law of Kepler

Let the planet move from P to Q, R to T and M to N in equal intervals of time t. Then according to the second law of Kepler we have area PSQ = area RST = area MSN.

Kepler's third law

The squares of the periodic times of the various planet are proportional to the cubes of the semi-major axes of their orbit.

Suppose $T_1, T_2, T_3 \ldots$ be the periodic times of several planets of semi major axes are $a_1, a_2, a_3 \ldots$ respectively. Then by third law

$$\frac{T_1^2}{a_1^3} = \frac{T_2^2}{a_2^3} = \frac{T_3^2}{a_3^3} \ldots$$

Theorem :- By assuming Kepler's law of planetary motion deduce Newton's law of gravitation.

Answer :- We know from Kepler's first law of planetary motion that a planet describes an ellipse having sun at one of its foci

Let the polar equation of ellipse ~~having the sun~~

$\frac{L}{r} = 1 + e\cos\theta$ where $e < 1$

But $\frac{1}{r} = u$

$\therefore Lu = 1 + e\cos\theta$ & $u = \frac{1}{L} + \frac{e\cos\theta}{L}$

$\therefore \frac{du}{d\theta} = -\frac{e}{L}\sin\theta$ and $\frac{d^2u}{d\theta^2} = -\frac{e}{L}\cos\theta$

From Kepler's second law of planetary motion, we know that the radius vector of the planet sweeps out equal area in equal intervals of time.

That is, the areal ~~vector~~ velocity of the planet is const.

$\therefore \frac{1}{2}r^2\frac{d\theta}{dt} = \text{Const}$
$= \frac{1}{2}h$ (Say)

where $P(r,\theta)$ and $Q(r+\delta r, \theta+\delta\theta)$ are positions of the planet at time t and $t+\delta t$ respectively.

But the transverse acceleration

$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$

$= \frac{1}{r}\cdot\frac{d}{dt}(h) = \frac{1}{r}\cdot 0 = 0$

This show that there is only radial acceleration. If P be the radial acceleration we know that

$P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$

$$= h^2u^2\left(\frac{1}{2} + \frac{e\cos\theta}{2} - \frac{e}{2}\cos\theta\right)$$

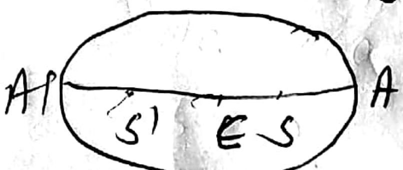
$$= \frac{h^2}{2}\cdot\frac{1}{r^2} = \frac{K}{r^2} \text{ where const } K = \frac{h^2}{2}$$

$$\therefore P \propto \frac{1}{r^2}$$

Hence the radial accelertion and therefore the force of attraction between the sun and the planet is inversly proportional to the square of the distance between them which is Newton's law of motion.

problem :— If v_1 and v_2 be the linear velocities of a planet when it is respectively nearest and farthest from the sun prove that $(1-e)v_1 = (1+e)v_2$

soln :— By Kepler's first law of planetary motion we know that each planet describes an ellipse having the sun is one of its foci. Let $AA' =$ the major axis $= 2a$ and C is the centre of ellipse.



we know that $CS = ae$
and $CA = CA' = a$

Here A is the nearest and A' is the farthest positions of the planet.

Now $SA = CA - CS = a - ae = a(1-e)$

and $SA' = CS + CA' = ae + a = a(1+e)$

In case of an ellipse we know that

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

when the planet is at A, then

$$v_1^2 = \mu\left(\frac{2}{SA} - \frac{1}{a}\right) = \mu\left(\frac{2}{a(1-e)} - \frac{1}{a}\right)$$

$$= \frac{\mu}{a} \cdot \frac{1+e}{1-e}$$

when the planet is A' then

$$v_2^2 = \mu\left(\frac{2}{SA'} - \frac{1}{a}\right)$$

$$= \mu\left(\frac{2}{a(1+e)} - \frac{1}{a}\right)$$

$$= \frac{\mu}{a} \cdot \frac{1-e}{1+e}$$

$$\therefore \frac{v_1^2}{v_2^2} = \frac{(1+e)^2}{(1-e)^2}$$

$$\therefore (1-e)v_1 = (1+e)v_2$$